

SOME INEQUALITIES FOR SINGULAR CONVOLUTION OPERATORS IN L^p -SPACES

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ABSTRACT. Suppose that a bounded function m satisfies a localized multiplier condition $\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} < \infty$, for some bump function ϕ . We show that under mild smoothness assumptions m is a Fourier multiplier in L^p . The approach uses the sharp maximal operator and Littlewood-Paley-theory. The method gives new results for lacunary maximal functions and for multipliers in Triebel-Lizorkin-spaces.

Introduction. Given a bounded function m the associated multiplier transformation T_m is defined by $[T_m f]^\wedge(\xi) = m(\xi) f^\wedge(\xi)$, $f \in \mathcal{S}(\mathbf{R}^n)$. Here $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing C^∞ -functions and $\mathcal{F}f = f^\wedge$ the Fourier transform. m is called a Fourier multiplier in $L^p(\mathbf{R}^n)$ if T_m extends to a bounded operator in $L^p(\mathbf{R}^n)$; the multiplier norm $\|m\|_{M_p}$ equals the operator norm of T_m .

Suppose that ϕ is a radial C_0^∞ -function with compact support in $\mathbf{R}_0^n = \mathbf{R}^n \setminus \{0\}$ and suppose that

$$(1) \quad \|m\|_{\dot{M}_p} = \sup_{t>0} \|\phi m(t \cdot)\|_{M_p} < \infty.$$

The purpose of this paper is to find easily verified conditions that (1) implies $m \in M_p$. The condition $\|m\|_{\dot{M}_p} < \infty$ is satisfied if and only if m is a Fourier multiplier on the class of homogeneous Besov-spaces $\dot{B}_{sq}^p$ (see Peetre [14, p. 132]). In fact the space M_p can be characterized by $\dot{M}_p$; a theorem of Johnson [10] states that $m \in M_p$ if and only if $m(\cdot + y) \in \dot{M}_p$ for every $y \in \mathbf{R}^n$. However, a straightforward verification of this condition seems to be impossible for many singular convolution operators.

In some applications it is useful to replace the ordinary dilations $x \mapsto tx$ by anisotropic ones: $x \mapsto t^P x = \exp(P \log t)x$, where P is a real $n \times n$ -matrix with trace ν , the real parts of the eigenvalues being contained in (a_0, a_0^0) , $a_0 > 0$. Then we ask, under which conditions $\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} < \infty$ implies $m \in M_p$. Throughout this paper ϕ will always be chosen as in the following

DEFINITION. $\phi \in C_0^\infty(\mathbf{R}_0^n)$ satisfies a Tauber condition with respect to the dilations (t^P) if for every $x \neq 0$ there is a t_x such that $\phi(t_x^P x) \neq 0$.

Sometimes we need special bump functions of the following kind: Let $\rho \in C^\infty(\mathbf{R}_0^n)$ be a P -homogeneous distance function; this means that $\rho(t^P x) = t\rho(x)$,

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$x \in \mathbf{R}^n$, $t > 0$ and $\rho(x) > 0$, $x \neq 0$. Then we set $\phi = \phi_0 \circ \rho$, where

$$(2) \quad \phi_0 \in C_0^\infty(\mathbf{R}_+), \quad \text{supp } \phi_0 \subset \left(\frac{1}{2}, 2\right), \quad \sum_{k \in \mathbf{Z}} \phi_0(2^k s) = 1, \quad \text{all } s > 0.$$

We note that every P -homogeneous distance function satisfies a triangle inequality $\rho(x+y) \leq b[\rho(x) + \rho(y)]$, for some $b \geq 1$.

It is easily seen that the condition $\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} < \infty$ is independent of the special choice of ϕ . In fact, assume that $\phi, \tilde{\phi}$ are chosen as in the definition. By a compactness argument, there are $s_0, \dots, s_N > 0$ such that $\sum_{i=1}^N \phi^2(s_i^P x) > 0$ for all $x \in \text{supp } \tilde{\phi}$. Since $M_1 M_p \subset M_p$, we have

$$\begin{aligned} \|\tilde{\phi} m(t^P \cdot)\|_{M_p} &\leq c \sum_{i=0}^N \|\phi^2(s_i^P \cdot) m(t^P \cdot)\|_{M_p} \\ &\leq c \sum_{i=0}^N \left\| \phi(\cdot) m\left(\left(\frac{t}{s_i}\right)^P \cdot\right) \right\|_{M_p} \leq c \sup_{s>0} \|\phi m(s^P \cdot)\|_{M_p}. \end{aligned}$$

We are most interested in the cases $1 < p < \infty$. For $p = 1$ a satisfactory result is the Hörmander multiplier criterion [9]. Here the condition $\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_1} < \infty$ is replaced by the somewhat stronger assumption

$$(3) \quad \sup_{t>0} \int_{|x| \geq \omega} |\mathcal{F}^{-1}[\phi m(t^P \cdot)]| dx \leq B(1 + \omega)^{-\varepsilon}, \quad \text{all } \omega > 0.$$

(3) implies that T_m is of weak type $(1, 1)$ and $m \in M_p$, $1 < p < \infty$. The usual assumption

$$\sup_{t>0} \|\phi m(t^P \cdot)\|_{\mathcal{L}_\alpha^2} < \infty, \quad \alpha > n/2,$$

($\mathcal{L}_\alpha^2$ denoting the Bessel-potential space as in Stein [18]) implies (3) for some B , if $\varepsilon < \alpha - n/2$.

We use the following notations: $\mathcal{S}_0$ denotes the subspace of Schwartz functions whose Fourier transforms are compactly supported in $\mathbf{R}_0^n$. Δ_h is the difference operator, $\Delta_h f = f(\cdot + h) - f(\cdot)$. The Lipschitz space Λ_ε is normed by

$$\|f\|_{\Lambda_\varepsilon} = \|f\|_\infty + \sup_h |h|^{-\varepsilon} \|\Delta_h f\|_\infty, \quad \text{if } 0 < \varepsilon < 1.$$

By $|S|$ we denote the Lebesgue measure of a set S . The barred integral $\bar{f}_S$ denotes the mean value $|S|^{-1} \int_S f(y) dy$. c will be a general constant with different values in different occurrences.

1. Main result.

THEOREM 1. *Suppose that m is a bounded function which satisfies for some p , $1 < p < \infty$, $\varepsilon > 0$*

$$(i) \quad \sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} \leq A,$$

$$(ii) \quad \sup_{t>0} \int_{|x| \geq \omega} |\mathcal{F}^{-1}[\phi m(t^P \cdot)](x)| dx \leq B(1 + \omega)^{-\varepsilon}.$$

Then

$$\|m\|_{M_p} \leq cA[\log(2 + B/A)]^{|1/p-1/2|}.$$

REMARK. Of course, condition (ii) alone implies $m \in M_p$, $1 < p < \infty$, with multiplier norm $\leq cB$, which may however be much larger than the constant in the theorem. This constant is actually sharp; it cannot be replaced by $A[\log(2 + B/A)]^\gamma$ with $\gamma < |1/p - 1/2|$. This can be seen by a well-known counterexample of Littman, McCarthy, Rivière [12], modified in Triebel's monograph [19]. Choose ϕ as in (2) and vectors σ_k , satisfying $\rho(\sigma_k) = (2b)^k$. Define

$$m_N(\xi) = \sum_{k=N}^{2N} e^{i\sigma_k \cdot \xi} \phi(\xi - \sigma_k).$$

Since $\|\phi m_N(A_t \cdot)\|_{M_p} \leq c$ and $\|D^\alpha[\phi m_N(A_t \cdot)]\|_{M_p} \leq c2^{Na^\alpha|\alpha|}$ for all multi-indices α , Theorem 1 implies $\|m_N\|_{M_p} \leq c_p N^{\gamma(p)}$, with $\gamma(p) = |1/p - 1/2|$.

On the other hand, the discussion in [19, p. 125] shows that $\|m_N\|_{M_p} \geq c'_p N^{\gamma(p)}$.

The counterexample shows that the condition (1) alone does not imply $m \in M_p$. In the following corollaries we shall see that this is valid under weak smoothness assumptions on m . The proof of Theorem 1 is given in §2.

COROLLARY 1. Suppose that for some $1 < p < \infty$

$$(i) \quad \sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} \leq A_0,$$

$$(ii) \quad \sup_{t>0} \int_{|h| \leq 2^{-t}} \|\Delta_h[\phi m(t^P \cdot)]\|_{M_p} dh \leq A_l.$$

Then

$$\|m\|_{M_p} \leq A_0 + \sum_{l>1} l^{|1/p-1/2|} A_l.$$

PROOF. We may choose ϕ as in (2). Let ψ be a C^∞ -function, supported in $\{\rho(\xi) \leq (8b)^{-1}\}$, $\int \psi(\xi) d\xi = 1$. Further set $\psi_l = 2^{ln} \psi(2^l \cdot)$, $\chi_l = \psi_l - \psi_{l-1}$ ($l \geq 1$), $\chi_0 = \psi_0$.

We split

$$\begin{aligned} m &= \sum_{j \in \mathbf{Z}} \phi(2^{-jP} \cdot) m \\ &= \sum_{j \in \mathbf{Z}} \sum_{l \geq 0} [\chi_l * (\phi m(2^{jP} \cdot))](2^{-jP} \cdot) =: \sum_{l \geq 0} m_l. \end{aligned}$$

Set $g_j = \phi m(2^j \cdot)$. Then $\chi_l * g_j$ is supported in $\{\frac{1}{4} \leq \rho(\xi) \leq 4\}$. If $l \geq 1$, we have for $2^k \leq s \leq 2^{k+1}$ (δ denoting Dirac measure)

$$\begin{aligned} \|\phi m(s^P \cdot)\|_{M_p} &\leq c \sum_{j=k-4}^{k+4} \|\chi_l * g_j\|_{M_p} \\ &\leq c \sum_{j=k-4}^{k+4} \|(\delta - \psi_{l-1}) * g_j + (\psi_l - \delta) * g_j\|_{M_p} \\ &\leq c \sum_{j=k-4}^{k+4} \left[\int |\psi_l(\eta)| \|\Delta_\eta g_j\|_{M_p} d\eta + \int |\psi_{l-1}(\eta)| \|\Delta_\eta g_j\|_{M_p} d\eta \right] \\ &\leq c(A_{l-1} + A_l). \end{aligned}$$

For all multi-indices α it follows by a similar computation ($2^k \leq s \leq 2^{k+1}$)

$$\begin{aligned} \|D^\alpha(\phi m_l(s^P \cdot))\|_2 &\leq c \sum_{j=k-4}^{k+4} \sum_{\beta \leq \alpha} \int |\psi_l^{(\beta)}(\eta)| \|g_j(\xi - 2^{-l}\eta) - g_j(\xi - 2^{-l+1}\eta)\|_\infty d\eta \\ &\leq c 2^{l|\alpha|} (A_{l-1} + A_l). \end{aligned}$$

Now we apply Theorem 1 and obtain

$$\|m_l\|_{M_p} \leq c l^{1/p-1/2} (A_{l-1} + A_l), \quad l \geq 1.$$

Analogously $\|m_0\|_{M_p} \leq c A_0$, and the assertion follows by summation.

COROLLARY 2. Suppose that $\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} < \infty$, for some $p \in (1, \infty)$.

(i) If for some $\varepsilon > 0$

$$\sup_{t>0} \sup_{h \in \mathbf{R}^n} |h|^{-\varepsilon} \|\Delta_h[\phi m(t^P \cdot)]\|_{M_p} < \infty$$

then $m \in M_p$.

(ii) If $\sup_{t>0} \|\phi m(t^P \cdot)\|_{\Lambda_\varepsilon} < \infty$, then $m \in M_r$, $|1/r - 1/2| < |1/p - 1/2|$.

PROOF. (i) is weaker than the assertion of Corollary 1. (ii) then follows by interpolating the inequalities

$$\|\Delta_h[\phi m(t^P \cdot)]\|_{M_p} \leq c, \quad \|\Delta_h[\phi m(t^P \cdot)]\|_{M_2} \leq c|h|^\varepsilon.$$

2. Proof of Theorem 1.

2.1. Some tools needed in the proof. Let r be a distance function, homogeneous with respect to the adjoint dilations t^{P^*} , satisfying a triangle inequality with constant b . Let $\mathscr{W}$ be the collection of all r -balls

$$Q = \{x; r(x_0 - x) \leq 2^k\}, \quad x_0 \in \mathbf{R}^n, \quad k \in \mathbf{Z},$$

x_0 is the “center” of Q , $2^k = \text{rad } Q$ its “radius”.

The Hardy-Littlewood maximal operator with respect to $\mathscr{W}$ is defined for functions with values in a Banach-space B by

$$\mathscr{M}f(x) = \sup_{x \in Q \in \mathscr{W}} \int_Q |f(y)|_B dy.$$

By $f^\#$ we denote the Fefferman-Stein sharp maximal function, defined by

$$f^\#(x) = \sup_{x \in Q \in \mathcal{W}} \int_Q |f(y) - f|_B dy.$$

The basic fact about $f^\#$ is

PROPOSITION. Assume that $1 < p < \infty$, $1 \leq p_0 \leq p$ and $f \in L^{p_0}(\mathbf{R}^n, B)$. If $f^\# \in L^p(\mathbf{R}^n)$, then $\mathcal{M}f \in L^p(\mathbf{R}^n)$ and $\|\mathcal{M}f\|_p \leq c\|f^\#\|_p$.

The proof is an adaptation of the proof given by Fefferman and Stein [8] in the more general setting of homogeneous spaces (see [15]). Another tool needed in the proof is Littlewood-Paley theory [18, 13]. Let $\phi \in C_0^\infty(\mathbf{R}_0^n)$ and $\eta_k = \mathcal{F}^{-1}[\phi(2^{-kP} \cdot)]$, $g(f) = (\sum_{k \in \mathbf{Z}} |\eta_k * f|^2)^{1/2}$. Then $\|g(f)\|_p \leq c\|f\|_p$, $1 < p < \infty$. We will choose $\phi = \Phi_0 \circ \rho$ as in (1). Then we also have $\|f\|_p \leq c\|g(f)\|_p$, $1 < p < \infty$. Let $\tilde{\phi} \in C_0^\infty(\mathbf{R}_0^n)$ be equal to 1 on $\text{supp } \phi$. Then we associate to $\tilde{\phi}$ in the same way the functions $\tilde{\eta}_k$ and $\tilde{g}(f)$.

2.2. Proof of the theorem. By duality we may assume $2 \leq p < \infty$. We associate to $T = T_m$ a vector-valued operator $\tilde{T}$, defined by $[\tilde{T}f]_k = \eta_k * Tf$. We will show that

$$(4) \quad \|(\tilde{T}f)^\#\|_p \leq cAN^{1/2-1/p}\|f\|_p$$

where

$$N = \max(\varepsilon^{-1}, a_0^{-1}) \log_2(2 + B/A).$$

If $f \in \mathcal{S}_0$, $\tilde{T}f$ is a priori in $L^p(l^2)$. By Littlewood-Paley theory and the Fefferman-Stein inequality we get

$$\begin{aligned} \|Tf\|_p &\leq c_1\|g(Tf)\|_p = c_1\|\tilde{T}f\|_{L^p(l^2)} \\ &\leq c_1\|\mathcal{M}(\tilde{T}f)\|_p \leq c_2\|(\tilde{T}f)^\#\|_p \leq c_3AN^{1/2-1/p}\|f\|_p. \end{aligned}$$

It remains to prove (4). In order to apply interpolation arguments it is useful to linearize the operator $f \mapsto (\tilde{T}f)^\#$. Fix $f \in L^p$. Following [8, p. 157] we may find for each $x \in \mathbf{R}^n$ a ball $Q_x \in \mathcal{W}$ containing x , the center and the radius being measurable functions of x , further functions $\chi_k(x, y)$, with $(\sum |\chi_k(x, y)|^2)^{1/2} \leq 1$, $x \in \mathbf{R}^n$, $y \in Q_x$, such that the following inequality holds:

$$(\tilde{T}f)^\#(x) \leq 2Sf(x)$$

where

$$(5) \quad Sf(x) = \int_{Q_x} \sum \left[\eta_k * Tf(y) - \int_{Q_x} \eta_k * Tf(z) dz \right] \chi_k(x, y) dy.$$

Define $l(x)$ by $\text{rad } Q_x = 2^{l(x)}$. Instead of S we consider the following operators σ_1 , σ_2 acting on sequence-valued functions $F = \{f_k\}$, $H = \{h_k\}$.

$$(6) \quad \sigma_1(F, x) = \int_{Q_x} \sum_{|k+l(x)| \leq N} \left[\tilde{\eta}_k * f_k(y) - \int \tilde{\eta}_k * f_k \right] \chi_k(x, y) dy,$$

$$(7) \quad \sigma_2(H, x) = \int_{Q_x} \sum_{|k+l(x)| > N} \left[\eta_k * Th_k(y) - \int \eta_k * Th_k \right] \chi_k(x, y) dy.$$

In 2.3 and 2.4 we will show that

$$(8) \quad \|\sigma_1(F)\|_p \leq cN^{1/2-1/p}\|F\|_{L^p(l^p)}$$

and

$$(9) \quad \|\sigma_2(H)\|_p \leq cA\|H\|_{L^p(l^2)},$$

the constant c being independent of A , N and the choice of Q_x , $\chi_k(x, y)$. We proceed by observing

$$Sf = \sigma_1(\{\eta_k * Tf\}) + \sigma_2(\{\tilde{\eta}_k * f\}).$$

By Littlewood-Paley theory (9) implies

$$\|\sigma_2(\{\tilde{\eta}_k * f\})\|_p \leq cA\|f\|_p.$$

Using the hypothesis (i) we get

$$\begin{aligned} \|\{\eta_k * Tf\}\|_{L^p(l^p)}^p &= \sum \|\eta_k * T(\tilde{\eta}_k * f)\|_p^p \leq A^p \sum \|\tilde{\eta}_k * f\|_p^p \\ &\leq A^p \|\{\eta_k * f\}\|_{L^p(l^2)}^p \leq cA^p \|f\|_p^p \end{aligned}$$

and from (8) we conclude

$$\|\sigma_1\{\eta_k * Tf\}\|_p \leq cAN^{1/2-1/p}\|f\|_p.$$

These estimates imply (4).

2.3. Estimation of $\sigma_1(F)$. Since $\sigma_1(F) \leq 2\mathcal{M}[(\sum_k |\tilde{\eta}_k * F_k|^2)^{1/2}]$, it follows by L^2 -boundedness of $\mathcal{M}$

$$\|\sigma_1(F)\|_2 \leq c \left(\sum_k \|\eta_k * F_k\|_2^2 \right)^{1/2} \leq c' \|F\|_{L^2(l^2)}.$$

If $p = \infty$, we have

$$\begin{aligned} \|\sigma_1(F)\|_\infty &\leq \left\| \mathcal{M} \left[\sup_{l \in \mathbf{Z}} \left(\sum_{|k+l| \leq N} |\eta_k * F_k|^2 \right)^{1/2} \right] \right\|_\infty \\ &\leq cN^{1/2} \sup_{k \in \mathbf{Z}} \|\eta_k * F_k\|_\infty \leq c' N^{1/2} \|F\|_{L^\infty(l^\infty)}. \end{aligned}$$

Now an application of the Riesz-Thorin interpolation theorem establishes (8).

2.4. Estimation of $\sigma_2(H)$. The operator $\sigma_2(H)$ represents the “remainder”-terms similarly treated as in the Calderón-Zygmund theory. By L^2 -boundedness of $\mathcal{M}$ and the Plancherel theorem we get

$$\begin{aligned} \|\sigma_2(H)\|_2^2 &= \sum_k \|\eta_k * Th_k\|_2^2 \\ &= \|\phi(2^{-kP} \cdot) m h_k^\wedge\|_2^2 \leq A^2 \sum_k \|h_k^\wedge\|_2^2 = A^2 \|H\|_{L^2(l^2)}^2. \end{aligned}$$

We show

$$(10) \quad \|\sigma_2(H)\|_\infty \leq cA\|H\|_{L^\infty(l^2)}$$

and (9) follows by interpolation.

We need a further splitting of σ_2 . Denote by R_x the ball with same center as Q_x and $\text{rad } R_x = 2b \text{ rad } Q_x$. For a function H we denote by $R_x H$ multiplication with the indicator function of R_x ; similarly define $R_x^c H$ for the complement R_x^c . We have the majorization $\sigma_2(H, x) \leq I(x) + II(x)$, where

$$I(x) = \int_{Q_x} \left(\sum_k |\eta_k * T(R_x h_k)(y)|^2 \right)^{1/2} dy$$

and

$$II(x) = \int_{Q_x} \left(\sum_{|k+l(x)| > N} |\{\eta_k * T(R_x^c h_k)(y) - \int \eta_k * T(R_x^c h_k)(z) dz\}|^2 \right)^{1/2} dy.$$

By Hölder's inequality and Plancherel's theorem we get

$$\begin{aligned} |I(x)| &\leq |Q_x|^{-1/2} \left(\sum \|\eta_k * T(R_x h_k)\|_2^2 \right)^{1/2} \\ &\leq |Q_x|^{-1/2} A \left(\sum \|R_x h_k\|_2^2 \right)^{1/2} \\ &\leq cA \int_{R_x} \sum |h_k(y)|^2 dy \\ &\leq cA \|H\|_{L^\infty(I^2)}. \end{aligned}$$

To estimate $II(x)$ set $K_k(x) = \mathcal{F}^{-1}[\phi m(A_{2^k} \cdot)]$. Then with

$$E_k(x, y, z) = \int_{R_x^c} 2^{k\nu} |K_k(2^{kP^*}(y-w)) - K_k(2^{kP^*}(z-w))| dw,$$

it happens that

$$(11) \quad E_k(x, y, z) \leq cB \min\{2^{-\varepsilon(k+l(x))}, 2^{a_0(k+l(x))}\},$$

whenever $y, z \in Q_x$.

Summing a geometrical series we obtain

$$\begin{aligned} |II(x)| &\leq \sup_{y, z \in Q_x} \left(\sum_{|k+l(x)| > N} [E_k(x, y, z)]^2 \right)^{1/2} \|H\|_{L^\infty(I^\infty)} \\ &\leq cB \max\{2^{-\varepsilon N}, 2^{-a_0 N}\} \|H\|_{L^\infty(I^\infty)} \\ &\leq cA \|H\|_{L^\infty(I^\infty)} \leq cA \|H\|_{L^\infty(I^2)}. \end{aligned}$$

(11) follows by a standard calculation. Denote by x_0 the center of R_x . Then for $w \in R_x^c$, $y \in Q_x$

$$r(y-w) \geq r(x_0-w)/b - r(x_0-y) \geq 2^{l(x)};$$

hence

$$E_k(x, y, z) \leq 2 \int_{r(u) \geq 2^{l(x)}} 2^{k\nu} |K_k(2^{kP^*} u)| du \leq cB 2^{-(k+l(x))\varepsilon}$$

by hypothesis (ii). If $k + l(x) < -N$, we use the fact that $\tilde{\phi} * K_k = K_k$ and obtain by Taylor's formula

$$\begin{aligned} E_k(x, y, z) &\leq 2 \int_0^1 \int_{R_x^\varepsilon} |2^{k\nu} [(2^{kP^*}(y-z) \cdot \nabla) K_k] (2^{kP^*}(z-w+sy-sz))| dz ds \\ &\leq c \| [2^{kP^*}(y-z) \cdot \nabla] \tilde{\phi} * K_k \|_1 \\ &\leq c \| [2^{kP^*}(y-z) \cdot \nabla] \tilde{\phi} \|_1 \| K_k \|_1 \\ &\leq c B 2^{(k+l(x))a_0}, \end{aligned}$$

if $y, z \in R_x$.

This completes the estimation of $\sigma_2(H)$ and concludes the proof of the theorem.

3. Some variants and applications.

3.1. *The case $p = 1$.* There is a simpler counterpart of Theorem 1.1 for $p = 1$ which strengthens slightly the Hörmander multiplier criterion. It involves a weak-type $(1, 1)$ and an (H^1, L^1) -estimate for the operator T_m . Here H^1 is the parabolic Hardy-space, defined as in [1] with respect to the (t^{P^*}) -dilations.

THEOREM 2. *Suppose that the hypotheses (i), (ii) of Theorem 1 are satisfied with $p = 1$. Then*

$$(a) \quad \|T_m f\|_1 \leq cA [\log(2 + B/A)]^{1/2} \|f\|_{H^1}.$$

$$(b) \quad \sup_{\alpha > 0} \alpha |\{|T_m f| > \alpha\}| \leq cA \log \left(2 + \frac{B}{A} \right) \|f\|_1.$$

PROOF. To prove (a) we use the atomic decomposition (see Latter and Uchiyama [11]). Let a be an atom, supported in $\{r(x_0 - x) \leq 2^l\}$, $\|a\|_\infty \leq c2^{-l\nu}$. Choose N as in the proof of Theorem 1 and split $Ta = T_{l,1}a + T_{l,2}a$, where $T_{l,1}a = \sum_{|k+l|\leq N} \eta_k * Ta$. Using the standard Calderón-Zygmund estimates it follows

$$\|T_{l,2}a\|_1 \leq cB \max(2^{-\varepsilon N}, 2^{-a_0 N}) \|a\|_1 \leq cA.$$

Further

$$\begin{aligned} \|T_{l,1}a\| &\leq A \left\| \sum_{|k+l|\leq N} |\eta_k * a| \right\|_1 \\ &\leq cAN^{1/2} \left\| \left(\sum_k |\eta_k * a|^2 \right)^{1/2} \right\|_1 \leq c'AN^{1/2}, \end{aligned}$$

by Littlewood-Paley theory in H^1 .

The proof of (b) is similar and involves a Calderón-Zygmund decomposition. We can only achieve the larger constant cAN , because Littlewood-Paley functions do not define bounded operators in L^1 . $\square$

REMARK. The counterexample m_N mentioned in §1 shows that the constants in Theorem 2 are sharp. For the (H^1, L^1) -estimate this follows from [19, p. 125]. The essential part of the kernel $\mathcal{F}^{-1}[m_N]$ lies near the points σ_k , $N \leq k \leq 2N$, and a straightforward computation shows that $\|\mathcal{F}^{-1}[m_N]\|_{L^{1,\infty}} \geq cN$ ($L^{1,\infty}$ denotes the Lorentz-space). Let $\chi \in \mathcal{S}$, $\hat{\chi}(\xi) = 1$ near 0, $\chi_l = 2^{l\nu} \chi(2^{lP^*} \cdot)$. For large l

we have $T_{m_N} \chi_I = \mathcal{F}^{-1}[m_N]$; this implies $\|T_{m_N}\|_{(L^1, L^{1\infty})} \geq cN$ for the weak-type operator-“norm” of T_{m_N} .

3.2. Application to quasiradial multipliers.

COROLLARY 3. *Let $\rho \in C^\infty(\mathbf{R}_0^n)$ be a P -homogeneous-distance function and $m = m_0 \circ \rho$, where $m_0 \in L^\infty(\mathbf{R}_+)$. Suppose that for some p , $1 \leq p < 2n/(n+1)$,*

$$\sup_{t>0} \|\phi m_0 \circ t\rho\|_{M_p} < \infty.$$

Then $m_0 \circ \rho \in M_r$, $p < r \leq 2$.

PROOF. The smoothness assumption of Corollary 2, (ii) is satisfied, since the necessary conditions for quasiradial multipliers [17] imply

$$\sup_{t>0} \|\phi_0 m_0(t \cdot)\|_{B_{\alpha p}^{p'}} \leq c \sup_{t>0} \|\phi m_0 \circ t\rho\|_{M_p},$$

$\alpha = (n-1)(1/p - 1/2)$. Here $\phi_0 \in C_0^\infty(\mathbf{R}_+)$ and $B_{\alpha p}^{p'}(\mathbf{R})$ is the standard Besov space defined in [19]. Now $B_{\alpha p}^{p'} \subset \Lambda_\varepsilon$ if $0 < \varepsilon < \alpha - 1/p'$ and $\alpha - 1/p' > 0$ if $p < 2n/(n+1)$. The assertion follows from Corollary 2 and the elementary inequality

$$\|\phi_0 \circ \rho m_0 \circ t\rho\|_{\Lambda_\varepsilon(\mathbf{R}^n)} \leq c \|\phi_0 m_0(t \cdot)\|_{\Lambda_\varepsilon(\mathbf{R})}. \quad \square$$

The following criterion for quasiradial multipliers is proved in [16].

COROLLARY 4. *Let $\rho \in C^\infty(\mathbf{R}_0^n)$, the unit sphere $\{\rho(\xi) = 1\}$ being strictly convex. Then*

$$\|m_0 \circ \rho\|_{M_p} \leq c \sup_{t>0} \|\phi_0 m_0(t \cdot)\|_{L_\alpha^2(\mathbf{R})},$$

$$\alpha > n|1/p - 1/2|, \quad 1 < p \leq 2(n+1)/(n+3).$$

The condition $\sup_{t>0} \|\phi m \circ t\rho\|_{M_p} < \infty$ can be verified following Stein's treatment of the Bochner-Riesz multiplier [7, 16]. The approach via Corollary 2 considerably simplifies the proof of Corollary 4 in [16]. It avoids also the weighted norm inequality in Christ's proof of (essentially) the same result (see [3]).

3.3. Lacunary maximal operators. Given a multiplier m , we define for $f \in \mathcal{S}$ the lacunary maximal operator T_m^* by

$$T_m^* f = \sup_{k \in \mathbf{Z}} |\mathcal{F}^{-1}[m(2^{kP} \cdot) f^\wedge]|.$$

To prove boundedness results for T_m we shall need information about a vector-valued singular integral operator τ , defined for functions $F = \{F_{k,l}\}$ with values in $l^2(\mathbf{Z}^2)$ by

$$[\tau(F)]_k = \sum_l \eta_{k+l} * F_{k,l}.$$

LEMMA. $\|\tau(F)\|_{L^p(l^2(\mathbf{Z}^2))} \leq c \|F\|_{L^p(l^2(\mathbf{Z}^2))}$, $1 < p < \infty$.

PROOF. For $p = 2$ the inequality follows by Plancherel's theorem. Then for $p < 2$, by Calderón-Zygmund theory we are led to verify the following weak Hörmander condition

$$\int_{r(x) \geq 2bt} \left[\sum_k \left| \sum_l [\eta_{k+l}(x-y) - \eta_{k+l}(x)] \alpha_{k,l} \right|^2 \right]^{1/2} dx \leq c \left(\sum_{k,l} \alpha_{k,l}^2 \right)^{1/2},$$

whenever $r(y) \leq t$, $(\alpha_{k,l}) \in l^2(\mathbf{Z}^2)$. The verification of (12) is a routine matter, so it is omitted. The case $p > 2$ follows by observing that the adjoint τ^* is similarly defined as τ (k, l are exchanged). $\square$

THEOREM 3. *Suppose that for some $1 < p < \infty$, $r = \min(p, 2)$, $\varepsilon > 0$*

$$(i) \quad \left(\int_0^\infty \|\phi m(t^P \cdot)\|_{M_p}^r \frac{dt}{t} \right)^{1/r} < \infty,$$

$$(ii) \quad \left(\int_0^\infty \left[\sup_h |h|^{-\varepsilon} \|\Delta_h \phi m(t^P \cdot)\|_{M_p} \right]^r \frac{dt}{t} \right)^{1/r} < \infty.$$

Then

$$\left\| \left(\sum_k |\mathcal{F}^{-1}[m(2^{kP} \cdot) f^\wedge]|^2 \right)^{1/2} \right\|_p \leq c \|f\|_p.$$

PROOF. Choose ϕ as in (2),

$$a_l = \|\phi m(2^{lP} \cdot)\|_{M_p}, \quad b_l = \sup |h|^{-\varepsilon} \|\Delta_h [\phi m(2^{lP} \cdot)]\|_{M_p}.$$

Then the hypotheses of the theorem are equivalent with $\sum (a_l^r + b_l^r) < \infty$; this essentially requires the same argument as in the Introduction.

We apply the lemma with $F_{k,l} = \mathcal{F}^{-1}[\tilde{\phi}(2^{-(k+l)P} \cdot) m(2^{-kP} \cdot) f^\wedge]$ to deduce

$$\left\| \left(\sum_k |\mathcal{F}^{-1}[m(2^{-kP} \cdot) f^\wedge]|^2 \right)^{1/2} \right\|_p \leq \left\| \left(\sum_{k,l} |F_{k,l}|^2 \right)^{1/2} \right\|_p.$$

If $p > 2$ we have by Minkowski's inequality

$$\left\| \left(\sum_{k,l} |F_{k,l}|^2 \right)^{1/2} \right\|_p \leq \left(\sum_l \left\| \left(\sum_k |F_{k,l}|^2 \right)^{1/2} \right\|_p^2 \right)^{1/2},$$

whereas if $p < 2$ we use $l^p \subset l^2$ and interchange summation and integration to get

$$\left\| \left(\sum_{k,l} |F_{k,l}|^2 \right)^{1/2} \right\|_p \leq \left(\sum_l \left\| \left(\sum_k |F_{k,l}|^2 \right)^{1/2} \right\|_p^p \right)^{1/p}.$$

Denote by r_k the sequence of Rademacher functions (see [18, p. 276]) and let

$$m_{l,s} = \sum_k r_k(s) \tilde{\phi}(2^{-(k+l)P} \cdot) m(2^{-kP} \cdot), \quad s \in [0, 1].$$

An application of Corollary 1.2 gives

$$\|m_{l,s}\|_{M_p} \leq \sum_{j=-3}^3 a_{l+j} + b_{l+j}, \quad \text{uniformly in } s \in [0, 1].$$

By Chinchin's inequality and interchanging the order of integrals we see

$$\left\| \left(\sum_k |F_{k,l}|^2 \right)^{1/2} \right\|_p^p \leq c \int_0^1 \|m_{l,s}\|_{M_p}^p dx \leq c \sum_{j=-3}^3 a_{l+j}^p + b_{l+j}^p.$$

Now summation over l proves the assertion. $\square$

Of course, Theorem 3 implies boundedness of T_m^* in L^p . For $p > 2$ there is a simpler result which follows from Littlewood-Paley theory and does not rely on Theorem 1.

COROLLARY 5. Suppose $2 \leq p < \infty$ and

$$\left(\int_0^\infty \|\phi m(t^P \cdot)\|_{M_p}^2 \frac{dt}{t} \right)^{1/2} < \infty.$$

Then $\|T_m^* f\|_p \leq c \|f\|_p$.

PROOF. We use the inequality

$$\left\| \sum_l \eta_l * g_l \right\|_p \leq \left\| \left(\sum_l |g_l|^2 \right)^{1/2} \right\|_p,$$

$1 < p < \infty$, which, by duality, is a consequence of Littlewood-Paley theory. Now

$$\begin{aligned} \|T_m^* f\|_p &\leq \left\| \left(\sum_k |\mathcal{F}^{-1}[m(2^{-kP} \cdot) f^\wedge]|^p \right)^{1/p} \right\|_p \\ &\leq c \left(\sum_k \left\| \left(\sum_l |\eta_{k+l} * \mathcal{F}^{-1}[m(2^{-kP} \cdot) f^\wedge]|^2 \right)^{1/2} \right\|_p^p \right)^{1/2} \\ &\leq c \left(\sum_l \left[\sum_k \|\mathcal{F}^{-1}[\phi(2^{-(k+l)P} \cdot) m(2^{-kP} \cdot)] * \tilde{\eta}_{k+l} * f\|_p^p \right]^{2/p} \right)^{1/2} \\ &\leq c \left(\sum_l \|\phi m(2^{lP} \cdot)\|_{M_p}^2 \right)^{1/2} \left(\sum_k \|\tilde{\eta}_k * f\|_p^p \right)^{1/p}, \end{aligned}$$

and a second application of Littlewood-Paley theory implies the assertion.

COROLLARY 6. Suppose that $m \in M_p$ satisfies for some $\delta > 0$ $|m(\xi)| \leq c|\xi|^\delta$, if $|\xi| \leq 1$ and $|m(\xi)| \leq c|\xi|^{-\delta}$, if $|\xi| \geq 1$.

(i) If $p > 2$, then $\|T_m^* f\|_r \leq c_r \|f\|_r$, $2 \leq r < p$.

(ii) If $p < 2$, and $\sup_{t>0} \|\phi m(t^P \cdot)\|_{\Lambda_\epsilon} < \infty$, then $\|T_m^* f\|_r \leq c \|f\|_r$, $p < r \leq 2$.

The proof follows by interpolation. Note that (i) is already contained in [4].

REMARK. In many cases, the decay condition at the origin is not valid, but m is smooth near $\xi = 0$. Then one may split $m = m_0 + m_1$, where m_0 is compactly supported and smooth and equals m near the origin. $T_{m_0}^* f$ is majorized by the Hardy-Littlewood maximal function $\mathcal{M}f$, and $T_{m_1}^*$ can be handled by the above corollaries. For example we can deduce the following result of Duoandikoetxea and

Rubio de Francia [6] (which, however, does not require the full strength of Theorem 1.1):

Let μ be a compactly supported measure satisfying $\mu^\wedge(\xi) \leq c(1 + |\xi|)^{-\delta}$,

$$\left\| \sup_{k \in \mathbb{Z}} \left| \int f(x - 2^{kP^*} y) d\mu(y) \right| \right\|_p \leq c \|f\|_p, \quad 1 < p \leq \infty.$$

Write $\mu = m + \tilde{m}$, where $m(\xi) = 0$ near the origin and $\text{supp } \tilde{m}$ is compact. Since μ is compactly supported, $m, \tilde{m}$ are smooth; further $|D^\alpha \mu^\wedge(\xi)| \leq c(1 + |\xi|)^{-\delta}$ for every multi-index α . Then $|T_{\tilde{m}}^* f| \leq cMf$. If $t \geq 1$ we have

$$\|\phi m(t^P \cdot)\|_\infty \leq ct^{-\delta a_0}, \quad \|D^\alpha(\phi m(t^P \cdot))\|_\infty \leq ct^{a_0}, \quad |\alpha| = 1$$

which implies $\sup_{t>0} \|\phi m(t^P \cdot)\|_{\Lambda_\epsilon} < \epsilon$, for some $\epsilon > 0$. $\square$

3.4. *Multipliers on Triebel-Lizorkin spaces.* Define for $1 < p, q < \infty$, η_k as in 2.1,

$$g_q(f) = \left(\sum_{k=-\infty}^{\infty} |\eta_k * f|^q \right)^{1/q}$$

and the homogeneous Triebel-Lizorkin space $\dot{F}^{pq} = \dot{F}^{pq}(P)$ by $\|f\|_{\dot{F}^{pq}} = \|g_q(f)\|_p$.

$\dot{F}^{pq}$ should be considered as a subspace of $\mathcal{S}'(\mathbb{R}^n)$ modulo polynomials; the definition depends on the dilation group (t^P) .

Let $\dot{\mathcal{M}}_{pq} = \dot{\mathcal{M}}_{pq}(P)$ be the subspace of bounded functions whose norms

$$\|m\|_{\dot{\mathcal{M}}_{pq}} = \sup\{\|\mathcal{F}^{-1}[mf^\wedge]\|_{\dot{F}^{pq}}; f \in \mathcal{S}_0, \|f\|_{\dot{F}^{pq}} \leq 1\}$$

are finite. Note that $\mathcal{S}_0$ is dense in $\dot{F}^{pq}$. Multipliers in $\dot{F}^{pq}$ are multipliers in the whole scale $\dot{F}_s^{pq}$, $-\infty < s < \infty$ (defined in [19] for isotropic dilations) since $\dot{F}_s^{pq} = I_s \dot{F}^{pq}$, where $I_s f = \mathcal{F}^{-1}[\rho^{-s} f^\wedge]$ for some P -homogeneous distance function $\rho \in C^\infty(\mathbb{R}_0^n)$. For simple properties of $\dot{\mathcal{M}}_{pq}$ we refer to Triebel [19, p. 128], where the inhomogeneous case is discussed. Observe that $\dot{\mathcal{M}}_p = \dot{\mathcal{M}}_{pp}(P)$ equals the space of multipliers on anisotropic homogeneous Besov spaces $\dot{B}_{sq}^p$ as mentioned in the Introduction.

THEOREM 4. *Suppose that m is a bounded function satisfying for some p , $1 < p < \infty$, $\epsilon > 0$*

$$(i) \quad \sup_{t>0} \|\phi m(t^P \cdot)\|_{M_p} \leq A,$$

$$(ii) \quad \sup_{t>0} \int_{|x| \geq \omega} |\mathcal{F}^{-1}[\phi m(t^P \cdot)]| dx \leq cB(1 + \omega)^{-\epsilon}.$$

Then m is a Fourier multiplier in $\dot{F}^{pq}(P)$, $|1/q - 1/2| \leq |1/p - 1/2|$, and $\|m\|_{\dot{\mathcal{M}}_{pq}} \leq cA[\log(2 + B/A)]^{1/p-1/q}$.

PROOF. By duality, we may assume $p > q$. It suffices to consider the case $q = p'$; the remaining cases follow by interpolation. The proof is a repetition of the arguments needed for Theorem 1, so we omit the details. The operators S, σ_1, σ_2 are defined as in (5), (6), (7) but now with $(\sum |\chi_k(x, y)|^{p'})^{1/p'} \leq 1$. Then

$$\|\sigma_1(F)\|_p \leq cN^{1/p'-1/p} \|F\|_{L^p(\ell^p)}, \quad \|\sigma_2(H)\|_p \leq c\|A\|_{L^p(\ell^{p'})}.$$

Instead of Plancherel's theorem and Littlewood-Paley theory, we use the hypothesis $\sup_{t>0} \|\phi m(t^P \cdot)\|_{M_{p'}} \leq A$ and the definition of $\dot{F}^{pp'}$. $\square$

As in §1, this theorem implies several corollaries, e.g.

COROLLARY 7. *Suppose that for some $\varepsilon > 0$*

$$\sup_{t>0} \|\phi m(t^P \cdot)\|_{\Lambda_\varepsilon} < \infty.$$

If m is a multiplier on the homogeneous Besov space $\dot{F}^{pp}(P)$, then it is also a multiplier on $\dot{F}^{rs}(P)$, $p < r, s < p'$.

It is an interesting problem whether the hypothesis of Corollary 7 implies $m \in \mathcal{M}_{2s}(P)$ for some $|1/s - 1/2| > |1/p - 1/2|$.

During the preparation of this paper the author was informed by A. Carbery, that he also established some of the results of this paper (see [2]), using another approach. In particular he found Corollaries 2 and 6, as well as some weak-type estimates in the endpoint cases.

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